Int. J. Open Problems Complex Analysis, Vol. 13, No. 3, November 2021
ISSN 2074-2827; Copyright ©ICSRS Publication, 2021
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Certain Properties for Analytic and p-Valent Functions with Negative Coefficients Associated with Sãlãgean qth-Order Differential Operator

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Received 15 July 2021; Accepted 29 October 2021

Abstract

We introduce a class of analytic and p-valent functions with negative coefficients defined with Sãlãgean q th-order differential operator. By using coefficient estimates, we obtain distortion theorems and closure theorems. Also we obtain radii of closed to convex, starlikeness and convexity for these functions of this class. We also obtain the property of preserving integral operator defined with

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -p.$$

We also obtain sharp lower bounds for $Re\{\frac{f(z)}{f_m(z)}\}$, $Re\{\frac{f_m(z)}{f(z)}\}$, $Re\{\frac{f'(z)}{f'_m(z)}\}$ ***and*** $Re\{\frac{f'_m(z)}{f'(z)}\}$.

Keywords: *Analytic,* p-*Valent, coefficient estimates, Distortion theorem, Closure theorem, Integral operator, Partial sums.*
Mathematics Subject Classification 2020: 30C45.

1 Introduction

We will state several properties of the class of analytic and p-valent functions which is defined with [7] as follows

$$1 + \frac{1}{b}\{\frac{D^j f^q(z)}{D^i f^q(z)} - (p-q)^{j-i}\} \prec \frac{1+Az}{1+Bz} \tag{1.1}$$

$z \in \triangle := \{z : z \in \mathbb{C}.\ \ |z| < 1\}$; $-1 \leq B < A \leq 1;\ j > i;\ j \in \mathbb{N}$; $i, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $b \in \mathbb{C} - \{0\}$.
We use the symbol $S^{j,i}_{p,q}[A, B; b]$ of complex order b to denote the class of analytic and p-valent functions defined by (1.1) .

This work in this paper will help us to expand other important topics for this new class of functions later such as the Hankel determinant and other properties.

Definitions and Preliminaries

Let $\mathfrak{A}$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \qquad (p \in \mathbb{N} := \{1, 2, 3, ...\}), \tag{1.2}$$

which are analytic in the open unit disk $\triangle$.
Suppose $f^{(q)}$ denote the qth-order ordinary differential operator which is defined for a function $f \in \mathfrak{A}$ as follows

$$f^{(q)}(z) = \frac{p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} a_k z^{k-q},$$

where $p > q;\ p \in \mathbb{N};\ q \in \mathbb{N}_0,\ z \in \triangle$. Frasin [6] introduced the differential operator $D^m f^q(z)$ as follows:

$$D^m f^q(z) = \frac{(p-q)^m p!}{(p-q)!} z^{p-q} + \sum_{k=p+1}^{\infty} (k-q)^m \frac{k!}{(k-q)!} a_k z^{k-q}. \tag{1.3}$$

We notice from (1.3) that $D^0 f^0(z) = f(z), D^0 f^1(z) = zf^{'}(z)$ and $D^m f^{(0)}(z) = D^m f(z)$ is a known operator which is introduced by *Sãlãgean* [8].

Let $\mathfrak{T}$ denote the subclass of $\mathfrak{A}$ consisting functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \qquad (a_k \geq 0,\ p \in \mathbb{N}). \tag{1.4}$$

Clearly, we have the following relationships:

1. $S_{p,q}^{j,i}[1,-1;b] = S_{p,q}^{j,i}(b)$;
2. $S_{p,0}^{n+1,n}[1,-1;b] = S_n(p,b)$; (Akbulut et al.[1])
3. $S_{1,0}^{1,0}[1,-1;b] = S(b)$ $(b \in \mathbb{C}-\{0\})$; (Nasr and Aouf [7] and wiatrowski [12])
4. $S_{1,0}^{2,1}[1,-1;b] = K(b)$ $(b \in \mathbb{C}-\{0\})$; (Nasr and Aouf [7] and wiatrowski [12])
5. $S_{1,0}^{1,0}[1,-1;1-\alpha] = S^*(\alpha)$ $\;for\;$ $0 \leq \alpha < 1$, where $S^*(\alpha)$ denotes the class of starlike functions of order α in $\triangle$.

We define the class $T_{p,q}^{j,i}[A,B;b]$ as $T_{p,q}^{j,i}[A,B;b] = S_{p,q}^{j,i}[A,B;b] \cap \mathfrak{T}$.

In this paper, we will display the ratio of a function of the form (1.4) which belongs to the class $T_{p,q}^{j,i}[A,B;b]$ to its sequence of partial sums $f_m(z) = z^p - \sum_{k=p+1}^{m+p-1} a_k z^k$ when the coefficients of f are sufficiently small to obtain the coefficient estimates which are introduced in the following section. Also, We will determine sharp lower bounds for $Re\{\frac{f(z)}{f_m(z)}\}$, $Re\{\frac{f_m(z)}{f(z)}\}$, $Re\{\frac{f'(z)}{f'_m(z)}\}$ and $Re\{\frac{f'_m(z)}{f'(z)}\}$.

Sheil-Small [9] noticed that inf $Re\{\frac{f(z)}{f_m(z)}\}$ for a starlike function *f(z)* defined by (1.2) for $p=1$ occurs when $m=1$ and In [2], a sharp lower bound was obtained for $Re\{\frac{f(z)}{f_1(z)}$ when *f(z)* is convex of order σ $(0 \leq \sigma < 1)$. E. M. Silvia[11] investigated lower bounds on $Re\{\frac{f(z)}{f_m(z)}\}$ for *f(z)* defined by (1.4) with $p=1$ which is convex and starlike of order σ. She noticed that $Re\{\frac{f(z)}{f_m(z)}\}$ $\geq \frac{1}{(2-\sigma)}$ for *f(z)* defined by (1.4) with $p=1$ which is starlike of order σ and that $Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{(3-\sigma)}{(4-2\sigma)}$ which is convex of order σ.
These results are sharp with $m=1$.

Generally, lower bounds on ratios like $Re\{\frac{f(z)}{f_m(z)}\}$, $Re\{\frac{f_m(z)}{f(z)}\}$ have been determined to be sharp only when $m=1$. In this paper, we determine sharpness for all values of m. The lower bounds are strictly increasing functions of m.

2 Coefficient Estimates

In the current section, we will obtain the coefficient estimates inequality for the class $T_{p,q}^{j,i}[A,B;b]$ as follows.

Theorem 2.1. *Assume the function $f(z)$ be defined by (1.4). Then we notice that $f(z)$ is in the class $T_{p,q}^{j,i}[A,B;b]$ if and only if*

$$\sum_{k=p+1}^{\infty} L_k^{j,i} \frac{k!}{(k-q)!} a_k \le \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i, \tag{2.1}$$

where $L_k^{j,i} = (k-q)^i\{|b|(A-B)+(p-q)^{j-i}(1+|B|)\}+(k-q)^j(1+|B|)$.

The result is sharp with the extremal function

$$f(z) = z^p - \frac{|b|(A-B)(k-q)!p!}{L_k^{j,i}(p-q)!k!}(p-q)^i z^k.$$

Proof. *Suppose that the inequality (2.1) is satisfied . Then we have*

$$|D^j f^q(z) - (p-q)^{j-i} D^i f^q(z)| - |b(A-B)D^i f^q(z) - B\{D^j f^q(z) - (p-q)^{j-i} D^i f^q(z)\}|$$

$$= |\sum_{k=p+1}^{\infty} [(k-q)^j - (p-q)^{j-i}(k-q)^i] \frac{k!}{(k-q)!} a_k z^{k-q}| - |b(A-B)\frac{p!}{(p-q)!}(p-q)^i z^{p-q} -$$

$$\sum_{k=p+1}^{\infty} [b(A-B)(k-q)^i + B(p-q)^{j-i}(k-q)^i - B(k-q)^j] \frac{k!}{(k-q)!} a_k z^{k-q}|$$

$$\le \sum_{k=p+1}^{\infty} [(k-q)^i\{|b|(A-B)+(p-q)^{j-i}(1+|B|)\}+(k-q)^j(1+|B|)] \frac{k!}{(k-q)!} a_k$$

$$- \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i$$

$$\le 0, \quad by \quad inequality(2.1).$$

Conversely, suppose that

$$\left|\frac{D^j f^q(z) - (p-q)^{j-i} D^i f^q(z)}{b(A-B)D^i f^q(z) - B\{D^j f^q(z) - (p-q)^{j-i} D^i f^q(z)\}}\right|$$

$$= \left|\frac{\sum_{k=p+1}^{\infty}[(k-q)^i(p-q)^{j-i} - (k-q)^j]\frac{k!}{(k-q)!} a_k z^{k-q}}{\frac{b(A-B)p!}{(p-q)!}(p-q)^i z^{p-q}}\right.$$

$$\left.\frac{1}{-\sum_{k=p+1}^{\infty}\{(k-q)^i[b(A-B)+B(p-q)^{j-i}] - B(k-q)^j\}\frac{k!}{(k-q)!} a_k z^{k-q}}\right|$$

$$\le \frac{|\sum_{k=p+1}^{\infty}[(k-q)^i(p-q)^{j-i} + (k-q)^j]\frac{k!}{(k-q)!} a_k z^{k-q}|}{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i |z|^{p-q}}$$

$$\frac{1}{-\sum_{k=p+1}^{\infty}\{(k-q)^i[|b|(A-B)+|B|(p-q)^{j-i}] + |B|(k-q)^j \frac{k!}{(k-q)!}\} a_k |z|^{k-q}}$$

$$\le 1, \quad z \in \triangle.$$

As $Re\{z\} \leq |z|$ *for all* z*, we have*

$$Re\Big\{\frac{\sum_{k=p+1}^{\infty}[(k-q)^i(p-q)^{j-i}+(k-q)^j]\frac{k!}{(k-q)!}a_kz^{k-q}}{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i|z|^{p-q}-\sum_{k=p+1}^{\infty}\{(k-q)^i[|b|(A-B)+}$$

$$\frac{1}{|B|(p-q)^{j-i}]+|B|(k-q)^j\}\frac{k!}{(k-q)!}a_k|z^{k-q}|}\Big\} \tag{2.2}$$

$$\leq 1. \tag{2.3}$$

We suppose $z \to 1^-$ *through real values in (2.2), then we have*

$$\sum_{k=p+1}^{\infty}\{(k-q)^i[|b|(A-B)+(p-q)^{j-i}(1+|B|)]+(k-q)^j(1+|B|)\}\frac{k!}{(k-q)!}a_k \leq \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i \tag{2.4}$$

This concludes the required condition.

Finally, the function

$$f(z) = z^p - \frac{|b|(A-B)(k-q)!p!}{L_k^{j,i}(p-q)!k!}(p-q)^iz^k \qquad (k \geq p+1) \tag{2.5}$$

is an extremal function for the theorem.

Corollary 2.1. *Let the function* f(z) *defined by* (1.4) *be in the class* $T_{p,q}^{j,i}[A,B;b]$*. Then*

$$a_k \leq \frac{|b|(A-B)p!(k-q)!}{k!(p-q)!L_k^{j,i}}(p-q)^i \qquad (k \geq p+1). \tag{2.6}$$

The result is sharp for the function $f(z)$ *given by (2.5).*

3 Distortion Theorems

In this section, we will display the minimum and maximum values for $|f(z)|$ and $|f'(z)|$.

Theorem 3.1. *Let* $f(z)$ *defined by* (1.4) *be in the class* $T_{p,q}^{j,i}[A,B;b]$*. Then we have*

$$|z|^p-\frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^{p+1} \leq |f(z)| \leq |z|^p+\frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^{p+1} \tag{3.1}$$

for $z \in \triangle$. The result is sharp with the extremal function

$$f(z) = z^p - \frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^{p+1}. \tag{3.2}$$

Proof. *Since $f(z) \in T_{p,q}^{j,i}[A,B;b]$, with the aid of Theorem 2.1, we have*

$$\begin{aligned} L_{p+1}^{j,i}\frac{(p+1)!}{(p+1-q)!}\sum_{k=p+1}^{\infty} a_k &\leq \sum_{k=p+1}^{\infty} L_k^{j,i}\frac{k!}{(k-q)!}a_k \\ &\leq \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i, \end{aligned} \tag{3.3}$$

which implies that

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}. \tag{3.4}$$

So we can display that

$$\begin{aligned} |f(z)| &\geq |z|^p - |z|^{p+1}\sum_{k=p+1}^{\infty} a_k \\ &\geq |z|^p - \frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^{p+1} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |f(z)| &\leq |z|^p + |z|^{p+1}\sum_{k=p+1}^{\infty} a_k \\ &\leq |z|^p + \frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^{p+1} \end{aligned} \tag{3.6}$$

for $z \in \triangle$. Finally, the function

$$f(z) = z^p - \frac{|b|(A-B)}{(p+1)L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^{p+1}. \tag{3.7}$$

is an extremal function for the theorem.

Theorem 3.2. *Let $f(z)$ defined by (1.4) be in the class $T_{p,q}^{j,i}[A,B;b]$. Then we have*

$$p|z|^{p-1} - \frac{|b|(A-B)}{L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^p \leq |f'(z)| \leq p|z|^{p-1} + \frac{|b|(A-B)}{L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^p \tag{3.8}$$

for $z \in \triangle$. The result is sharp with the function in (3.2).
Proof. *With the aid of Theorem 2.1, we have*

$$\begin{aligned} L_{p+1}^{j,i}\frac{p!}{(p+1-q)!}\sum_{k=p+1}^{\infty} ka_k &\le \sum_{k=p+1}^{\infty} L_k^{j,i}\frac{k!}{(k-q)!}a_k \\ &\le \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i, \end{aligned} \tag{3.9}$$

which implies that

$$\sum_{k=p+1}^{\infty} ka_k \le \frac{|b|(A-B)}{L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}. \tag{3.10}$$

Hence, with the aid of (3.10), we have

$$\begin{aligned} |f'(z)| &\ge p|z|^{p-1} - |z|^p\sum_{k=p+1}^{\infty} ka_k \\ &\ge p|z|^{p-1} - \frac{|b|(A-B)}{L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^p. \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} |f'(z)| &\le p|z|^{p-1} + |z|^p\sum_{k=p+1}^{\infty} ka_k \\ &\le p|z|^{p-1} + \frac{|b|(A-B)}{L_{p+1}^{j,i}}(p-q)^i\{p+1-q\}|z|^p. \end{aligned} \tag{3.12}$$

for $z \in \triangle$. Finally the result is sharp with the function in (3.2).

4 Closure Theorems

We will display some results associating with the closure properties of functions which belong to $T_{p,q}^{j,i}[A,B;b]$.

Suppose the function $f_w(z)$ be defined as follows

$$f_w(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,w}z^k (a_{k,w} \ge 0) \tag{4.1}$$

for $z \in \triangle$ and $w = 1, 2, ..., m$.

Theorem 4.1. *Suppose the functions $f_w(z)$ defined by (4.1) be in the class $T_{p,q}^{j,i}[A,B;b]$ for every $w = 1,2,...,m$. Then the function $h(z)$ defined by*

$$h(z) = \sum_{w=1}^{m} c_w f_w(z) \qquad (c_w \geq 0) \tag{4.2}$$

is also in the same class $T_{p,q}^{j,i}[A,B;b]$, where

$$\sum_{w=1}^{m} c_w = 1. \tag{4.3}$$

Proof. *From the definition of $h(z)$, we have*

$$h(z) = z^p - \sum_{k=p+1}^{\infty} (\sum_{w=1}^{m} c_w a_{k,w}) z^k. \tag{4.4}$$

Since $f_w(z)$ is in $T_{p,q}^{j,i}[A,B;b]$, then, for every $w = 1,2,...,m$, we obtain

$$\sum_{k=p+1}^{\infty} L_k^{j,i} \frac{k!}{(k-q)!} a_{k,w} \leq \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i, \tag{4.5}$$

for every $w = 1,2,...,m$. We can obtain that

$$\begin{aligned} &\sum_{k=p+1}^{\infty} L_k^{j,i} \frac{k!}{(k-q)!} (\sum_{w=1}^{m} c_w a_{k,w}) \\ &= \sum_{w=1}^{m} c_w (\sum_{k=p+1}^{\infty} L_k^{j,i} \frac{k!}{(k-q)!} a_{k,w}) \\ &\leq (\sum_{w=1}^{m} c_w) \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i = \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i \end{aligned} \tag{4.6}$$

with the aid of (4.5). This shows that the function $h(z)$ is in the class $T_{p,q}^{j,i}[A,B;b]$ with the aid of Theorem 2.1.

Theorem 4.2. *Let $f_{2p} = z^p$ and*

$$f_{p+k}(z) = z^p - \frac{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i}{\frac{k!}{(k-q)!} L_k^{j,i}} z^k \qquad (k \geq p+1) \tag{4.7}$$

for $p \in \mathbb{N}$,$-1 \leq B < A \leq 1$. Then f(z) is in the class $T_{p,q}^{j,i}[A,B;b]$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=p}^{\infty} \lambda_{p+k} f_{p+k}(z), \tag{4.8}$$

where $\lambda_{p+k} \geq 0$ $(k \geq p)$ *and* $\sum_{k=p}^{\infty} \lambda_{p+k} = 1$.

Proof. *Let*

$$\begin{aligned} f(z) &= \sum_{k=p}^{\infty} \lambda_{p+k} f_{p+k}(z) \\ &= z^p - \sum_{k=p+1}^{\infty} \frac{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i}{\frac{k!}{(k-q)!} L_k^{j,i}} \lambda_{p+k} z^k. \end{aligned} \tag{4.9}$$

Then it leads to

$$\sum_{k=p+1}^{\infty} L_k^{j,i} \frac{k!}{(k-q)!} \frac{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i}{\frac{k!}{(k-q)!} L_k^{j,i}} \lambda_{p+k} \tag{4.10}$$

$$= \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i \sum_{k=p+1}^{\infty} \lambda_{p+k} \tag{4.11}$$

$$\leq \frac{|b|(A-B)p!}{(p-q)!}(p-q)^i. \tag{4.12}$$

So by Theorem 2.1, $f(z) \in T_{p,q}^{j,i}[A, B; b]$.

Conversely, Suppose that the function $f(z)$ *defined by (1.4) belongs to the class* $T_{p,q}^{j,i}[A, B; b]$. *Then*

$$a_k \leq \frac{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i}{\frac{k!}{(k-q)!} L_k^{j,i}} \qquad (k \geq p+1). \tag{4.13}$$

Setting

$$\lambda_{p+k} = \frac{\frac{k!}{(k-q)!} L_k^{j,i}}{\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i} a_k \tag{4.14}$$

and

$$\lambda_{2p} = 1 - \sum_{k=p+1}^{\infty} \lambda_{p+k}. \tag{4.15}$$

We can see that $f(z)$ *can be expressed in the form (4.8). This proves Theorem 4.2.*

5 Integral Operators

We will display some closure properties for functions which is defined with an integral operator associated with function $f(z)$ defined by (1.4) and belongs to the class $T_{p,q}^{j,i}[A, B; b]$.

Theorem 5.1. *Suppose the function* f(z) *be defined by (1.4) in the class* $T_{p,q}^{j,i}[A,B;b]$, *and assume* c *be a real number such that* $c>-p$. *Then the function F(z) defined by*

$$F(z)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt \tag{5.1}$$

also belongs to the class $T_{p,q}^{j,i}[A,B;b]$.

Proof. *From the representation of* F(z), *it leads to*

$$F(z)=z^p-\sum_{k=p+1}^{\infty}b_kz^k,$$

where $b_k=(\frac{c+p}{k+c})a_k$. *Therefore,*

$$\begin{aligned}\sum_{k=p+1}^{\infty}L_k^{j,i}\frac{k!}{(k-q)!}b_k&=\sum_{k=p+1}^{\infty}L_k^{j,i}\frac{k!}{(k-q)!}(\frac{c+p}{k+c})a_k\\ &\leq\sum_{k=p+1}^{\infty}L_k^{j,i}\frac{k!}{(k-q)!}a_k\leq\frac{|b|(A-B)p!}{(p-q)!}(p-q)^i,\end{aligned}$$

since $f(z)\in T_{p,q}^{j,i}[A,B;b]$. *Then, by Theorem 2.1,* $F(z)\in T_{p,q}^{j,i}[A,B;b]$.

Putting $c=1-p$ in Theorem 4.1, we get corollary 5.1 .

Corollary 5.1. *Let the function* $f(z)$ *defined by (1.4) be in the class* $T_{p,q}^{j,i}[A,B;b]$ *and* $F(z)$ *be defined by*

$$F(z)=\frac{1}{z^{1-p}}\int_0^z\frac{f(t)}{t^P}dt. \tag{5.2}$$

Then $F(z)\in T_{p,q}^{j,i}[A,B;b]$.

Theorem 5.2. *Let* c *be a real number such that* $c>-p$. *If* $F(z)\in T_{p,q}^{j,i}[A,B;b]$, *then the function* $f(z)$ *defined by (5.1) is* p-valent *in* $|z|<R_p^*$, *where*

$$R_p^*=\inf_k\{\frac{p(c+p)k!(p-q)!L_k^{j,i}}{k(k+c)p!(k-q)!|b|(A-B)(p-q)^i}\}^{\frac{1}{k-p}}\qquad(k\geq p+1). \tag{5.3}$$

The result is sharp.

Proof. *Let* $F(z)=z^p-\sum_{k=p+1}^{\infty}a_kz^k$. *It follows from (5.1) that*

$$\begin{aligned}f(z)&=\frac{z^{1-c}(z^cF(z))'}{(c+p)},(c>-p)\\ &=z^p-\sum_{k=p+1}^{\infty}(\frac{(k+c)}{(c+p)})a_kz^k.\end{aligned}$$

To prove the result, it is sufficient to contain that

$$|\frac{f'(z)}{z^{p-1}} - p| \le p \quad for \quad |z| < R_p^*.$$

Now

$$\begin{aligned} |\frac{f'(z)}{z^{p-1}} - p| &= | - \sum_{k=p+1}^{\infty} \frac{k(k+c)}{(c+p)} a_k z^{k-p}| \\ &\le \sum_{k=p+1}^{\infty} \frac{k(k+c)}{(c+p)} a_k |z|^{k-p}. \end{aligned}$$

Thus $|\frac{f'(z)}{z^{p-1}} - p| \le p$ *if*

$$\sum_{k=p+1}^{\infty} \frac{k(k+c)}{p(c+p)} a_k |z|^{k-p} \le 1. \tag{5.4}$$

But Theorem 2.1 *assures that*

$$\sum_{k=p+1}^{\infty} \frac{L_k^{j,i} \frac{k!}{(k-q)!} a_k}{\frac{p!}{(p-q)!}\{|b|(A-B)(p-q)^i\}} \le 1. \tag{5.5}$$

Then (5.4) *will be satisfied if*

$$\frac{k(k+c)}{p(c+p)} |z|^{k-p} \le \frac{L_k^{j,i} \frac{k!}{(k-q)!}}{\frac{p!}{(p-q)!}\{|b|(A-B)(p-q)^i\}} \qquad (k \ge p+1),$$

or if

$$|z| \le \{\frac{p(c+p)k!(p-q)! L_k^{j,i}}{k(k+c)p!(k-q)!\{|b|(A-B)(p-q)^i\}}\}^{\frac{1}{k-p}} \qquad (k \ge p+1). \tag{5.6}$$

The required result is contained now from (5.6). The result is sharp for the function

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \frac{p!(k-q)!(k+c)|b|(A-B)(p-q)^i}{k!(p-q)!(c+p) L_k^{j,i}} z^k \qquad (k \ge p+1). \tag{5.7}$$

6 Radii of closed to convex, Starlikeness and Convexity

In this section, we will obtain the radii of closed to convex, starlikeness and convexity for functions $f(z)$ which is defined by (1.4) and belongs to the class $T_{p,q}^{j,i}[A, B; b]$.

Theorem 6.1. *Let the function* $f(z)$ *defined by (1.4) be in the class* $T^{j,i}_{p,q}[A,B;b]$, *then* $f(z)$ *is p-valently close-to-convex of order* α $(0 \leq \alpha \leq p)$ *in* $|z| < r_1(A,B,p,|b|,\alpha)$ *where*

$$r_1(A,B,p,|b|,\alpha) = \inf_k [\frac{(p-\alpha)(p-q)!k!L^{j,i}_k}{k|b|(A-B)p!(k-q)!(p-q)^i}]^{\frac{1}{k-p}}. \tag{6.1}$$

The result is sharp with the extremal function $f(z)$ *given by* (2.5).

Proof. *We must obtain that* $|\frac{f'(z)}{z^{p-1}} - p| \leq p-\alpha$ *for* $|z| < r_1(A,B,p,|b|,\alpha)$. *We have*

$$|\frac{f'(z)}{z^{p-1}} - p| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}. \tag{6.2}$$

Then $|\frac{f'(z)}{z^{p-1}} - p| \leq p-\alpha$ *if*

$$\sum_{k=p+1}^{\infty} \frac{k a_k}{p-\alpha}|z|^{k-p} \leq 1. \tag{6.3}$$

Hence, with the aid of Theorem 2.1, (6.3) will be true if

$$\frac{k}{p-\alpha}|z|^{k-p} \leq \frac{(p-q)!k!L^{j,i}_k}{|b|(A-B)p!(k-q)!(p-q)^i}. \tag{6.4}$$

Or if

$$|z| \leq [\frac{(p-\alpha)(p-q)!k!L^{j,i}_k}{k|b|(A-B)p!(k-q)!(p-q)^i}]^{\frac{1}{k-p}}, \quad (k \geq p+1). \tag{6.5}$$

The theorem follows easily from (6.5).

Theorem 6.2. *Suppose that the function* $f(z)$ *defined by* (1.4) *be in the class* $T^{j,i}_{p,q}[A,B;b]$, *then* $f(z)$ *is p-valently starlike of order* α $(0 \leq \alpha \leq p)$ *in* $|z| < r_2(A,B,p,|b|,\alpha)$ *where*

$$r_2(A,B,p,|b|,\alpha) = \inf_k [\frac{(p-\alpha)(p-q)!k!L^{j,i}_k}{(k-\alpha)|b|(A-B)p!(k-q)!(p-q)^i}]^{\frac{1}{k-p}}. \tag{6.6}$$

The result is sharp with the extremal function f(z) *given by* (2.5).

Proof. *We must show that* $|\frac{zf'(z)}{f(z)} - p| \leq p-\alpha$ *for* $|z| < r_2(A,B,p,|b|,\alpha)$. *We have*

$$|\frac{zf'(z)}{f(z)} - p| \leq \frac{\sum_{k=p+1}^{\infty}(k-p)a_k|z|^{k-p}}{1-\sum_{k=p+1}^{\infty} a_k|z|^{k-p}}. \tag{6.7}$$

Then $|\frac{zf'(z)}{f(z)} - p| \leq p - \alpha$ *if*

$$\sum_{k=p+1}^{\infty} \frac{(k-\alpha)}{(p-\alpha)} a_k |z|^{k-p} \leq 1. \tag{6.8}$$

Hence, by Theorem 2.1, (6.8) will be true if

$$\sum_{k=p+1}^{\infty} \frac{(k-\alpha)}{(p-\alpha)} |z|^{k-p} \leq \frac{(p-q)!k!L_k^{j,i}}{|b|(A-B)p!(k-q)!(p-q)^i} \tag{6.9}$$

Or if

$$|z| \leq [\frac{(p-\alpha)(p-q)!k!L_k^{j,i}}{(k-\alpha)|b||(A-B)p!(k-q)!(p-q)^i}]^{\frac{1}{k-p}}, \quad (k \geq p+1). \tag{6.10}$$

The theorem comes easily from (6.10) .

Corollary 6.1. *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $T_{p,q}^{j,i}[A,B;b]$, *then* $f(z)$ *is p-valently convex of order* α $(0 \leq \alpha \leq p)$ *in* $|z| < r_3(A,B,p,|b|,\alpha)$ *where*

$$r_3(A,B,p,|b|,\alpha) = \inf_k [\frac{p(p-\alpha)(p-q)!k!L_k^{j,i}}{k(k-\alpha)|b|(A-B)p!(k-q)!(p-q)^i}]^{\frac{1}{k-p}}. \tag{6.11}$$

The result is sharp with the extremal function $f(z)$ *given by* (2.5).

7 Partial sums

In this section, we will apply methods used by Silverman [10] and Cho and Owa [3] to determine the ratio of a function of the form (1.4) which belongs to the class $T_{p,q}^{j,i}[A,B;b]$ to its sequence of partial sums Liu et al.[6] $f_m(z) = z^p - \sum_{k=p+1}^{m+p-1} a_k z^k$ when the coefficients of *f(z)* are sufficiently small to satisfy condition (2.1). We will determine sharp lower bounds for $Re\{\frac{f(z)}{f_m(z)}\}$, $Re\{\frac{f_m(z)}{f(z)}\}$, $Re\{\frac{f'(z)}{f'_m(z)}\}$ and $Re\{\frac{f'_m(z)}{f'(z)}\}$.

Theorem 7.1. *If* $f(z)$ *of the form* (1.4) *satisfies condition* (2.1), *then* $Re\{\frac{f(z)}{f_m(z)}\} \geq \frac{(v_1-v_2)}{v_1}$ *where* $z \in \triangle$*.We denote* $L_{p+m}^{j,i}(p+m)!(p-q)!$ *and* $|b|(A-B)p!(p-q)^i(p+m-q)!$ *with* v_1 *and* v_2 *respectively. The result is sharp for every* m *with extremal function* $f(z) = z^p - \frac{v_2}{v_1} z^{p+m}$.

Proof. *We may write*

$$
\begin{aligned}
&\frac{v_1}{v_2}\left[\frac{f(z)}{f_m(z)}-\frac{(v_1-v_2)}{v_1}\right]\\
&=\frac{1-\sum_{k=p+1}^{m+p-1}a_kz^{k-p}-\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_kz^{k-p}}{1-\sum_{k=p+1}^{m+p-1}a_kz^{k-p}}\\
&=\frac{1+A(z)}{1+B(z)}.
\end{aligned}
$$

Set $(1+A(z))/(1+B(z))=(1+w(z))/(1-w(z))$*, so that* $w(z)=(A(z)-B(z))/(2+A(z)+B(z))$.

Then

$$
w(z)=\frac{-\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_kz^{k-p}}{2-2\sum_{k=p+1}^{m+p-1}a_kz^{k-p}-\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_kz^{k-p}}
$$

and

$$
|w(z)|\le\frac{\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_k}{2-2\sum_{k=p+1}^{m+p-1}a_k-\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_k}.
$$

Since $|w(z)|\le 1$*, then we have* $(2\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_k)\le 2-2\sum_{k=p+1}^{m+p-1}a_k$*, which can be written as*

$$
\sum_{k=p+1}^{m+p-1}a_k+\sum_{k=m+p}^{\infty}\frac{v_1}{v_2}a_k\le 1. \tag{7.1}
$$

It suffices to show that the left hand side of (7.1) *is bounded above by* $\sum_{k=p+1}^{\infty}L_k^{j,i}\frac{(k)!(p-q)!}{|b|(A-B)p!(p-q)^i(k-q)!}a_k$*, which is equivalent to*

$$
\begin{aligned}
&\sum_{k=p+1}^{m+p-1}\frac{L_k^{j,i}(k)!(p-q)!(p+m-q)!-v_2(k-q)!}{v_2(k-q)!}a_k+\\
&\sum_{k=m+p}^{\infty}\frac{L_k^{j,i}(k)!(p-q)!(m+p-q)!-v_1(k-q)!}{v_2(k-q)!}a_k\\
&\ge 0.
\end{aligned}
$$

To notice that $f(z)=z^p-\frac{v_2}{v_1}z^{p+m}$ *gives the sharp result, we observe for* $z=$

$re^{2\pi i/m}$ *that*

$$\frac{f(z)}{f_m(z)} = 1 - (\frac{v_2}{v_1})z^m \to 1 - \frac{v_2}{v_1}$$
$$= \frac{v_1 - v_2}{v_1} \quad whenr \to 1^-.$$

Theorem 7.2. *If $f(z)$ of the form (1.4) satisfies condition (2.1), then $Re\{\frac{f_m(z)}{f(z)}\} \geq \frac{v_1}{(v_1+v_2)}$ where $z \in \triangle$. We denote $L^{j,i}_{p+m}(p+m)!(p-q)!$ and $|b|(A-B)p!(p-q)^i(p+m-q)!$ with v_1 and v_2 respectively. The result is sharp for every m, with extremal function $f(z) = z^p - \frac{v_2}{v_1}z^{p+m}$.*

Proof. *We may write*

$$\frac{(v_1+v_2)}{v_2}\left[\frac{f_m(z)}{f(z)} - \frac{v_1}{v_1+v_2}\right]$$
$$= \frac{1 - \sum_{k=p+1}^{m+p-1} a_k z^{k-p} + \sum_{k=m+p}^{\infty} \frac{v_1}{v_2} a_k z^{k-p}}{1 - \sum_{k=p+1}^{\infty} a_k z^{k-p}}$$
$$= \frac{1 + A(z)}{1 + B(z)}.$$

Set $(1 + A(z))/(1 + B(z)) = (1 + w(z))/(1 - w(z))$, so that $w(z) = (A(z) - B(z))/(2 + A(z) + B(z))$.

Then

$$w(z) = \frac{\sum_{k=m+p}^{\infty} \frac{(v_1+v_2)}{v_2} a_k z^{k-p}}{2 - 2\sum_{k=p+1}^{m+p-1} a_k z^{k-p} + \sum_{k=m+p}^{\infty} \frac{(v_1-v_2)}{v_2} a_k z^{k-p}}$$

and

$$|w(z)| \leq \frac{\sum_{k=m+p}^{\infty} \frac{(v_1+v_2)}{v_2} a_k}{2 - 2\sum_{k=p+1}^{m+p-1} a_k - \sum_{k=m+p}^{\infty} \frac{(v_1+v_2)}{v_2} a_k}.$$

Since we have $|w(z)| \leq 1$, then $2\sum_{k=m+p}^{\infty} \frac{(v_1+v_2)}{v_2} a_k) \leq 2 - 2\sum_{k=p+1}^{m+p-1} a_k$, which can be written as

$$\sum_{k=m+p}^{\infty} \frac{v_1+v_2}{v_2} a_k + \sum_{k=p+1}^{m+p-1} a_k \leq 1. \tag{7.2}$$

Since the left hand side of (7.2) is bounded above by $\sum_{k=p+1}^{\infty} L^{j,i}_k \frac{(k)!(p-q)!}{|b|(A-B)p!(k-q)!(p-q)^i} a_k$, that ends the proof.

Theorem 7.3. *If $f(z)$ of the form (1.4) satisfies condition (2.1), then for $z \in \triangle$,*
(a) $Re\{\frac{f'(z)}{f'_m(z)}\} \geq \frac{(x_1-x_2)}{x_1}$.
(b) $Re\{\frac{f'_m(z)}{f'(z)}\} \geq \frac{x_1}{x_1+x_2}$. We denote $\frac{v_1}{p+m}$ and $\frac{v_2}{p}$ with x_1 and x_2 respectively. In both cases, the extremal function is $f(z) = z^p - \frac{v_2}{v_1} z^{p+m}$.

Proof. *We prove only (a), which is the same as the proof of Theorem 7.1. The proof of (b) follows the pattern of that in Theorem 7.2.*

We write

$$\frac{x_1}{x_2}\left[\frac{f'(z)}{f'_m(z)} - \frac{(x_1-x_2)}{x_1}\right]$$
$$= \frac{1 - \sum_{k=p+1}^{p+m-1} \frac{k}{p} a_k z^{k-p} - \sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k z^{k-p}}{1 - \sum_{k=p+1}^{p+m-1} \frac{k}{p} a_k z^{k-p}}$$
$$= \frac{1+A(z)}{1+B(z)}.$$

Set $(1+A(z))/(1+B(z)) = (1+w(z))/(1-w(z))$, so $w(z) = (A(z)-B(z))/(2+A(z)+B(z))$.

Then

$$w(z) = \frac{-\sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k z^{k-p}}{2 - 2\sum_{k=p+1}^{p+m-1} \frac{k}{p} a_k z^{k-p} - \sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k z^{k-p}}$$

and

$$|w(z)| \leq \frac{\sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k}{2 - 2\sum_{k=p+1}^{p+m-1} \frac{k}{p} a_k - \sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k}.$$

Since we have $|w(z)| \leq 1$, then $2\sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k \leq 2 - 2\sum_{k=p+1}^{p+m-1} \frac{k}{p} a_k$, which can be written as

$$\sum_{k=p+m}^{\infty} \frac{kx_1}{px_2} a_k + \sum_{k=p+1}^{p+m-1} \frac{k}{p} a_k \leq 1. \tag{7.3}$$

Since the left hand side of (7.3) is bounded above by $\sum_{k=p+1}^{\infty} L_k^{j,i} \frac{(k)!(p-q)!}{|b|(A-B)p!(p-q)^i(k-q)!} a_k$, the proof is complete.

8 Conclusion

We obtained coefficient estimates for the class $T^{j,i}_{p,q}[A, B; b]$. We used these coefficient estimates to obtain distortion theorems and closure theorems. Also we obtained radii of closed to convex, starlikeness and convexity for the class. We also obtained class preserving integral operator of the form

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t)dt, \quad c > -p.$$

Also, we determined sharp lower bounds for $Re\{\frac{f(z)}{f_m(z)}\}$, $Re\{\frac{f_m(z)}{f(z)}\}$, $Re\{\frac{f'(z)}{f'_m(z)}\}$ and $Re\{\frac{f'_m(z)}{f'(z)}\}$.

9 Open Problems

- The authors suggest to determine partial sums for meromrphic and p-valent functions of the form$f(z) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} a_k z^k$.

- The authors suggest to determine the Hankel determinant for the class $T^{j,i}_{p,q}[A, B; b]$.
- The authors suggest investigating the class $T_{p,q}$ with the recent differential operators in $[1, 2]$.

References

[1] M. Al-Kaseasbeh, M. Darus, and S. Al-Kaseasbeb, Certain differential sandwich theorem involved constructed differential operator. *International Information Institute (Tokyo). Information*, 19(10)(2016),4663-4670.

[2] M. Al-Kaseasbeh, M. Darus, Uniformly geometric functions involving constructed operators. *Journal of Complex Analysis*, 2017(ID 5916805), 1-7.

[3] S. Akbulut and E. Kadioğlu and M. Özdemir, On the subclass of p-valently functions, *J. Appl. Math. Comput.*, 147(1)(2004), 89-96.

[4] L. Brickman, D. J. Hallenbeck, T. H. MacGregor and D. R. Wilken, Convex hulls and extreme points of families of starlike and convex mappings, *Trans. Amer. Math. Soc.*, 185(1973), 413-428.

[5] N. E. Cho and S. Owa, Partial sums of certain meromorphic functions, *J. Inequal. Pure Appl. Math.*, 5(2) (2004), 1-7.

[6] B. A. Frasin, Neighborhoods of certain multivalent functions with negative coefficients, *Appl. Math. comput.*, 193(2007), 1-6.

[7] P. Goswami and M. K. Aouf, Majorization properties for certain classes of analytic functions using the *Sãlãgean* operator, *J. Appl. Math. lett.*, 23(2010), 1351-1354.

[8] J. L. Liu, R. Srivastava and Y. H. Rekha, *Integral transforms and partial sums of certain p-valent starlike functions, Revista de la Real Academia de Ciencias Exactas*, Físicas y Naturales. Serie A. Matemáticas, 113 (2) (2019), 845-859.

[9] M. A. Nasr and M. K. Aouf, Starlike functions of complex order, *J. Nature. Sci.Math.*, 25(1985), 1-12.

[10] G. S. *Sãlãgean*, *Subclasses of univalent functions*, Springer-Verlag, Berlin, 1983, 362-372.

[11] T. Sheil-Small, A note on the partial sums of convex schlicht functions, *Bull. London Math. Soc.*, 2(1970), 165-168.

[12] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.*, 209 (1997), 221-227.

[13] E. M. Silvia, On partial sums of convex functions of order. *Houston J. Math.*, 11(3)(1985), 397-404.

[14] P. Wiatrowski, *On the coefficients of some family of holomorphic functions*, Zeszyty Nauk. Uniw. Lodz Nauk. Mat.-Przyrod., 39(1970), 75-85.